Dynamical Systems
Tutorial 7: Linear Systems

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The material in this tutorial is taken from chapter 2 in Meiss.

A linear differential equation is an equation of the form

$$\frac{dx}{dt} = Ax$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.

As we saw in the previous tutorial, we are interested in linear differential equations as their behavior determines the stability of orbits of more general, nonlinear systems.

1 $2-$D Linear Systems

Recall, for a two-dimensional system, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if we denote $\tau \equiv \mathrm{tr}(A)$, $\delta \equiv \det(A)$, we have seen the different types of fixed points one may get (see Figure 2.1 from Meiss).

What happens in the general, $n-$dimensional case?

2 $n-$D Linear Systems

$$\dot{x}(t) = Ax,\ t \in \mathbb{R}, x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n} \tag{1}$$

For a 1-dimensional system, we saw the solution for a linear equation is the exponent. Let us check if $x(t) = e^{\mathbf{A}t}x_0$, is a solution:

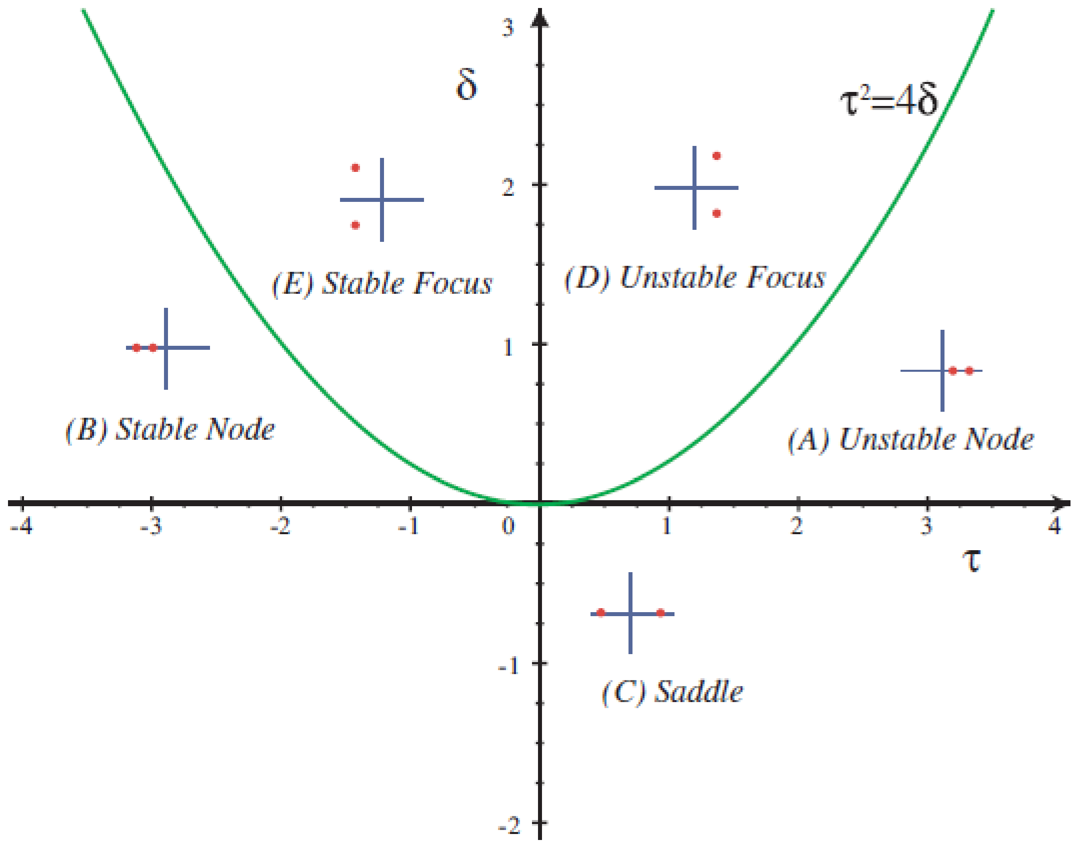

Figure 2.1. *Classification of the eigenvalues for a 2×2 linear system in the parameter space of the trace, τ, and determinant, δ.*

$$
\begin{aligned}
&\tfrac{d}{dt}(x(t)) = \tfrac{d}{dt}\left([I + At + \tfrac{1}{2!}A^2t^2 + \cdots + \tfrac{1}{n!}A^nt^n + \ldots]x_0\right) \\
&= [A + A^2t + \cdots + \tfrac{1}{(n-1)!}A^nt^{n-1} + \ldots]x_0 \\
&= A[I + At + \tfrac{1}{2!}A^2t^2 + \cdots + \tfrac{1}{(n-1)!}A^{n-1}t^{n-1} + \ldots]x_0 = Ae^{\mathbf{A}t}x_0 \\
&= Ax.
\end{aligned} \tag{2}
$$

How does the exponential $e^{\mathbf{A}t}$ behave in time? For this we recall the following notions from linear algebra.

2.1 Eigenvalues and eigenvectors

An eigenvector v is a nonzero solution to the equation

$$Av = \lambda v$$

for an eigenvalue λ. This equation has a solution only when the matrix $A - \lambda I$ is singular, or equivalently the characteristic polynomial (eq. 2.3 in Meiss) $p(\lambda) \equiv \det(\lambda I - A) = 0$. Notice v may be scaled to our convenience.

The characteristic polynomial (2.3) is an nth-order polynomial and so it has n zeros, λ_i. Some of the zeros may be identical, but these should be counted with their

▷ *algebraic multiplicity*: If a polynomial can be written $p(r) = (r - \lambda)^k q(r)$, with $q(\lambda) \neq 0$, then λ is a root with algebraic multiplicity k.

An eigenvalue whose algebraic multiplicity is larger than one is called a *multiple eigenvalue*. The fundamental theorem of algebra states that an nth degree polynomial has exactly n zeros when they are counted with their algebraic multiplicity.

Each eigenvector corresponds to a simple solution of the ODE: assume that $x(t) = c(t)v$ for $c : \mathbb{R} \to \mathbb{R}$ and substitute this into (2.1) to obtain

$$\dot{c}v = cAv = c\lambda v,$$

which when $v \neq 0$ implies that $\dot{c} = \lambda c$ since the eigenvector is constant. The general solution of this scalar ODE is $c(t) = e^{\lambda t} c_o$ for an arbitrary constant c_o. Therefore, the vector

$$x(t) = c_o e^{\lambda t} v \tag{2.4}$$

is a solution to (2.1). Geometrically, (2.4) corresponds to a straight-line solution (when λ is real): $x(t)$ is a vector along v whose length changes exponentially with time.

Since the equation is linear, the solutions obey the linear superposition principle. Hence any linear combination of solutions is a solution. This implies that the set of solutions of a linear ODE is a vector space. As a consequence, if there are k different eigenvector solutions of the form (2.4), then there is a more general solution of the form

$$x(t) = \Sigma_{i=1}^{k} c_i e^{\lambda_i t} v_i$$

for any value of the constants c_i.

Example: Consider the 2×2 system

$$\dot{x} = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} x . \tag{2.5}$$

The characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, so there are two eigenvalues, each with algebraic multiplicity one, $\lambda_1 = 2$ and $\lambda_2 = -3$. The eigenvector equations (2.2) are

$$(A - 2I)v_1 = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

$$(A + 3I)v_2 = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives the two solutions

$$x_1 = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad x_2 = c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \blacksquare$$

When $k = n$, or in other words the span of the eigenvectors is $\mathbb{R}^n$, then A is said to have a complete set of eigenvectors. In this case, the linear combination of solutions as above covers the entire solution space. However, sometimes matrices have multiple eigenvalues. Such an eigenvalue may have more than one eigenvector, though it need not. The number of its eigenvectors is called the

> ▷ *geometric multiplicity*: An eigenvalue λ has geometric multiplicity k if it has k linearly independent eigenvectors v_i, i.e., $(A - \lambda I)v_i = 0$ and $\dim(\mathrm{span}\{v_1, v_2, \ldots, v_k\}) = k$.

A basic theorem of linear algebra states that the geometric multiplicity of λ is at most its algebraic multiplicity. There are n independent eigenvectors when the geometric multiplicity of each eigenvalue equals its algebraic multiplicity; otherwise there is a deficiency of eigenvectors. What determines the structure of the solution of a linear ODE is the number of linearly independent eigenvectors A has and their associated eigenvalues.

2.2 Diagonalization

The simplest case is when A is *diagonalizable*, or *semisimple*. (n-distinct eigenvalues, real symmetric[1], hermitian[2]), that is, has n eigenvectors. We have $\Lambda = P^{-1}AP$, with Λ diagonal matrix, and P an invertible matrix comprised of A's eigenvectors.

So instead of solving:

$$\dot{x} = Ax = P\Lambda P^{-1}x \rightarrow P^{-1}\dot{x} = \Lambda P^{-1}x. \tag{3}$$

So $y = P^{-1}x$ we get:

$$\dot{y} = \Lambda y. \tag{4}$$

where now, $e^{\Lambda} = diag\{e^{\lambda_1}, \cdots, e^{\lambda_n}\}$. So the solution is (in vector form):

$$y(t) = e^{t\Lambda}c$$

Where c is a vector of coefficients determined by the initial conditions. To return to x, we note that $x_0 = Py_0 = Pc$, and so

$$x(t; x_o) = Pe^{t\Lambda}P^{-1}x_0 = e^{tA}x_0$$

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[1] $A = A^T$

[2] $A = A^H$

2.3 Complex eigenvalues

First, note that if the matrix A is real, then so are the coefficients of the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. Therefore, if $p(\lambda)$ has a complex root $\lambda = a + ib$, then its conjugate $\bar{\lambda} = a - ib$ is also a root. Moreover, if $Av = \lambda v$, then $A\bar{v} = \overline{\lambda v} = \bar{\lambda}\bar{v}$. Therefore, the corresponding eigenvectors are also complex conjugates.

Example: For the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the eigenvalues are $\lambda = \pm i$ and the eigenvectors are $v = \left(\begin{smallmatrix} 1 \\ \pm i \end{smallmatrix}\right)$. Choosing $P = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ and using (2.15) and property (iv) of Corollary 2.2 gives

$$e^{tA} = Pe^{t\Lambda}P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

which is the same as the real matrix, (2.29), obtained using the infinite series. ∎

Suppose that the $n \times n$ real matrix A has a complex eigenvector v and eigenvalue λ. These can be written in terms of their real and imaginary parts as

$$\lambda = a + ib, \quad v = u + iw.$$

Since $Av = \lambda v = (au - bw) + i(aw + bu)$ and A is real, then

$$Au = au - bw, \quad Aw = bu + aw.$$

If we let $P = [u, w]$ be the $n \times 2$ matrix with real columns u and w, then these two equations can be combined to obtain

$$AP = P\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{2.34}$$

giving a "normal form" that is not diagonal but relatively simple, as

$$e^{tB} = e^{taI}e^{tb\sigma} = e^{at}\begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}. \quad ∎ \tag{2.31}$$

for $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, I the unit matrix, and $\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The most general basis for solutions with complex eigenvalues is $t^k e^{\alpha t}\cos(\beta t)$ and $t^m e^{\alpha t}\sin(\beta t)$, where k and l are determined by the size of the jordan block (less than the maximal block).

2.4 Jordan form

For the general case we have *Jordan form*: $A = TJT^{-1}$, so $\dot{X} = AX$ means $X(t) = X_0 e^{At} = X_0 e^{TJT^{-1}t} = X_0 T e^{Jt} T^{-1}$.

An example of a Jordan cell 3×3:

$$\tilde{J} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$J = \lambda I + N$, N - nilpotent matrix ($\exists q \in \mathbb{N}$, s.t $N^q = 0_{n\times n}, q < n$, where N, J are $n \times n$ matrices).

The computation of the exponent operator on A is aided in two things:

1. we can use the binomial formula for convenience of representation, since $IN = NI$:

$$J^p = (\lambda I + N)^p = \big|_{p=2} (\lambda I + N)^2 = \lambda^2 I + N^2 + IN + NI$$

2. The fact that N is a *nilpotent* matrix assure us that the representation is finite.

so a Jordan form solution is:

$$e^{t\tilde{J}} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2!}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

Notice we get polynomial growth!

The subspace associated with a Jordan block is called the

▷ *generalized eigenspace*: Suppose λ_k is an eigenvector of a linear operator T with algebraic multiplicity n_k. The generalized eigenspace of λ_k is

$$E_k \equiv \ker\left[(T - \lambda_k I)^{n_k}\right]. \tag{2.35}$$

The theorem of primary decomposition (from linear algebra) states that the space spanned by the collection of all (invariant) generalized eigenspaces is the full space.

We define:

▷ *generalized eigenvector*: A nonzero solution to $(T-\lambda_j I)^{n_j} v = 0$, where n_j is the algebraic multiplicity of λ_j, is a generalized eigenvector of T.

These are the column vectors of the matrix T (the Jordanizing matrix).

2.5 The exponential

Finally, we can bring all of the above together by using the *semisimple-nilpotent decomposition*:

Theorem 2.8. *A matrix A on a complex vector space E, has a unique decomposition, $A = S + N$, where S is semisimple, N is nilpotent, and $[S, N] = 0$.*

where $[S,N] = SN - NS$, i.e. the matrices commute.

The semisimple-nilpotent decomposition leads to a compact and relatively computable formula for the exponential. Letting $A = S + N$, where $S = P\Lambda P^{-1}$, since N is nilpotent,

$$e^{tA} = e^{tS}e^{tN} = Pe^{t\Lambda}P^{-1}\left(\sum_{j=0}^{n-1}\frac{(tN)^j}{j!}\right). \tag{2.38}$$

notice $N^n = 0$ necessarily, and so the sum is indeed finite.

3 Spectral stability

▷ *spectral stability*: A linear system is spectrally stable if none of its eigenvalues has a positive real part.

The sign of the real part of the eigenvalue distinguishes the subspaces on which the solutions have growing or decaying behavior. Denote the (complex) generalized eigenvectors by $v_j = u_j + i w_j$. Then

▷ $E^u = \text{span}\{u_i, w_i : \text{Re}(\lambda_i) > 0\}$ is the *unstable* subspace,

▷ $E^c = \text{span}\{u_i, w_i : \text{Re}(\lambda_i) = 0\}$ is the *center* subspace, and

▷ $E^s = \text{span}\{u_i, w_i : \text{Re}(\lambda_i) < 0\}$ is the *stable* subspace.

Note that by the theorem of primary decomposition, $E = E^u \bigoplus E^c \bigoplus E^s$. Moreover, since each of the generalized eigenspaces is invariant, so are the stable, center and unstable subspaces. One can describe the evolution in each subspace by constructing a "restriction" to this subspace - see example. A system with no center subspace is called *hyperbolic*.

While finding whether or not a system is spectrally stable is relatively easy, this does not necessarily give information on stronger types of stability (such as linear stability). When the nilpotent part of A is non-zero, ie. we have eigenvalues whose geometric multiplicity is strictly smaller than their algebraic multiplicity, then e^{tA} contains terms that are powers of t multiplied by exponents. If $Im(\lambda) \geq 0$, these terms are unbounded.

4 Examples

Example: Consider the multiplicity-two case

$$A = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad p(\lambda) = (\lambda+1)^2(\lambda-1) = 0. \tag{2.40}$$

The eigenspace for $\lambda_3 = 1$ is obtained by solving

$$(A - I)v_3 = \begin{pmatrix} -2 & 1 & -2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix} v_3 = 0, \text{ thus } v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

To obtain the generalized eigenspace, for $\lambda_1 = \lambda_2 = -1$, solve

$$(A + I)^2\, v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{pmatrix} v = 0, \text{ thus } v = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix},$$

for arbitrary constants a and b. The space E_1 is spanned by $v_1 = (1, 0, 0)^T$ and $v_2 = (0, 1, 0)^T$. Setting $P = [v_1, v_2, v_3]$ gives

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$S = P\Lambda P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = A - S = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $N^2 = 0$, the final answer is

$$e^{tA} = Pe^{t\Lambda}P^{-1}e^{tN} = P \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} P^{-1}(I + tN),$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -2e^{-t} + 2e^{t} \\ 0 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & t & -2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & -2e^{-t} + 2e^{t} \\ 0 & 0 & e^{t} \end{pmatrix}. \quad \blacksquare$$

Example: Consider again the example (2.40). The eigenvalue $\lambda_3 = 1$ has eigenvector $v_3 = (0, 2, 1)^T$. Consequently the matrix $U = A|_{E^u}$ is the 1×1 matrix defined by the equation

$$Av_3 = 1v_3 = v_3 U,$$

and $U = (1)$. The dynamics restricted to this subspace is simply $\dot{c}_3 = 1c_3$. The stable subspace with eigenvalue $\lambda_1 = -1$ has basis $v_1 = (1, 0, 0)^T$ and $v_2 = (0, 1, 0)^T$, so that the stable matrix $S = A|_{E^s}$ is the 2×2 matrix defined by

$$A \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} U,$$

which gives $U = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. The dynamics in this subspace is therefore

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ -c_2 \end{pmatrix},$$

and $c_1,\ c_2$ are simply the x_1 and x_2 components. ■

Example: A matrix is *normal* if it commutes with its adjoint: $[A^*, A] = 0$, where $A^* = \bar{A}^T$ is the conjugate transpose of A. It is not hard to see that the eigenspaces of a normal matrix are orthogonal. The dynamics of a stable linear system with a nonnormal matrix can exhibit a surprising temporary growth. Consider, for example,

$$\dot{x} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix} x \quad \Rightarrow \quad x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -10 \\ 1 \end{pmatrix}. \tag{4.14}$$

The general solution shows that every initial condition is attracted to the origin, so the origin should be stable. However, points that start in the disk of radius δ about the origin can leave, at least for a while. For example, setting $c_1 = 9,\ c_2 = 1$, then $x(0) = (-1, 1)$. However, the second eigenvector quickly decays, leaving a large horizontal component. Consequently, the orbit can move away from the origin for some time, as shown in Figure 4.6.

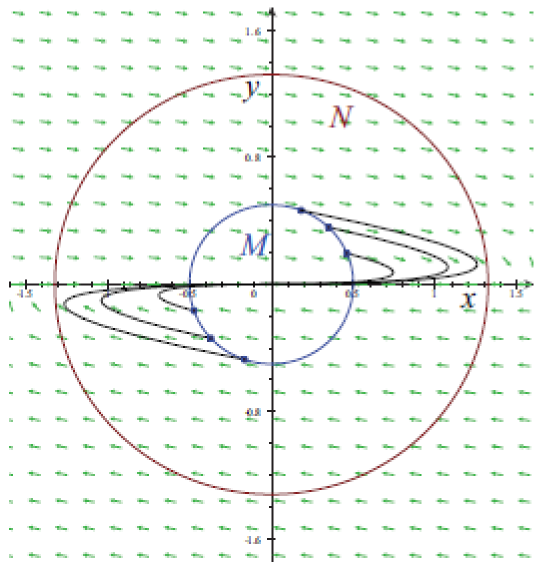

Figure 4.6. *Orbits of the system* (4.14) *that start in a neighborhood M never leave N.*

Bibliography

Meiss, J. D. (2007). Differential dynamical systems (Vol. 14). Siam.